$\mathbf{W}_\infty$ GRAVITY – A GEOMETRIC APPROACH

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A brief review is given of an adaptation of the coadjoint orbit method appropriate for the study of models with infinite-dimensional symmetry groups. It is illustrated on several examples, including derivation of the WZNW action of induced $D = 2$ $(N, 0)$ supergravity. As the main application, we present the geometric action on a generic coadjoint orbit of the deformed group of area-preserving diffeomorphisms. This action is precisely the anomalous effective WZNW action of $D = 2$ matter fields coupled to a chiral $\mathbf{W}_\infty$ gravity background. Similar actions are given which produce the KP hierarchy as on-shell equations of motion.

In Memoriam M. C. Polivanov

For the international scientific community Professor M. C. Polivanov was a highly esteemed theoretical physicist in a world-renowned institution where the foundations of numerous branches of modern theoretical and mathematical physics ranging from axiomatic quantum field theory [1] to soliton theory and quantum groups [2] took shape. But for the frequent foreign visitors to the Steklov Mathematical Institute he was much more. We shall always remember with deep admiration his profound and all-embracing erudition, invariably accompanied by the noble human warmth of a genuine Russian aristocrat.

1. INTRODUCTION

In the last few years, much attention has been devoted to the infinite-dimensional Lie algebra $\mathbf{W}_\infty$ and its generalizations $\mathbf{W}_{1+\infty}$ etc. [3, 4, 5]. These algebras are nontrivial "large N" limits of the associative, but non-Lie, conformal $\mathbf{W}_N$ algebras [6]. From a purely algebraic point of view, $\mathbf{W}_\infty$ is isomorphic to the centrally extended Lie algebra $\widetilde{\mathcal{DOP}}(S^1)$ of differential operators of arbitrary order on the circle [7, 8]. Geometrically, it is a nontrivial deformation of the Lie algebra $\mathbf{w}_\infty$ of area-preserving diffeomorphisms on a cylinder.

$\mathbf{W}_{(1)+\infty}$ algebras arise in various problems of two-dimensional physics: self-dual gravity [9], the first Hamiltonian structure of integrable Kadomtsev–Petviashvili (KP) hierarchy [10, 11], string field actions in the collective field theory approach [12], stringy black holes [13], conformal affine Toda theories [14]. One of the most remarkable manifestations of $\mathbf{W}_\infty$-type algebras is the recent discovery of a subalgebra of their "classical" limit $\mathbf{w}_\infty$ in $c = 1$ string theory as a symmetry algebra of special discrete states [15] or as the algebra of infinitesimal deformations of the ground ring [16].

A characteristic feature of all conformal field theory models, including those based on $\mathbf{W}_{(1)+\infty}$ algebras, is that their dynamics is entirely described in terms of an underlying infinite-dimensional Noether symmetry algebra containing the Virasoro algebra as a subalgebra. Thus, to study them it is natural to invoke [17] the general theory of group coadjoint orbits [18], extending it to the infinite-dimensional case.

In Section 2 below we briefly outline a general group coadjoint orbit formalism appropriate for studying geometric actions and their symmetries for arbitrary infinite-dimensional groups with central extensions.

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In Section 3, a solution of Ward identities for quantum effective (WZNW) actions in terms of geometric actions on group coadjoint orbits is discussed. As nontrivial examples, we present the explicit form of the geometric actions associated with the $D=2$ $(N,0)$ super-Virasoro group (for any $0 \leq N \leq 4$), which are the anomalous effective actions of induced $D=2$ $(N,0)$ supergravity. In Section 4 the group coadjoint orbit formalism is applied to derive the WZNW effective action of induced $\mathbf{W}_\infty$ gravity. Also, Lagrangian actions are given which yield the KP hierarchy as equations of motion.

2. BRIEF REVIEW OF COADJOINT ORBITS AND GEOMETRIC ACTIONS

2.1. Basic Ingredients. Let us consider an arbitrary (infinite-dimensional) group G with a Lie algebra $\mathcal{G}$ and its dual space $\mathcal{G}^*$. The adjoint and coadjoint actions of G and $\mathcal{G}$ on $\mathcal{G}$ and $\mathcal{G}^*$ are given by $\mathrm{Ad}(g)X = gXg^{-1}$, $\mathrm{ad}(X)Y = [X,Y]$ and $\langle \mathrm{Ad}^*(g)U \mid X\rangle = \langle U \mid \mathrm{Ad}(g^{-1})X\rangle$, $\langle \mathrm{ad}^*(X)U \mid Y\rangle = -\langle U \mid [X,Y]\rangle$. Here $g \in G$ and $X,\ Y \in \mathcal{G}$, $U \in \mathcal{G}^*$ are arbitrary elements, whereas $\langle\cdot \mid \cdot\rangle$ indicates the natural bilinear form "pairing" $\mathcal{G}$ and $\mathcal{G}^*$.

For theoretical physics applications, the primary interest lies in infinite-dimensional Lie algebras with a central extension $\widetilde{\mathcal{G}} = \mathcal{G} \oplus \mathbb{R}$ of $\mathcal{G}$ and, correspondingly, an extension $\widetilde{\mathcal{G}}^* = \mathcal{G}^* \oplus \mathbb{R}$ of the dual space $\mathcal{G}^*$. The central extension is given by a linear operator $\widehat{s}\colon \mathcal{G} \to \mathcal{G}^*$ satisfying

$$\widehat{s}\big([X,Y]\big) = \mathrm{ad}^*(X)\,\widehat{s}(Y) - \mathrm{ad}^*(Y)\,\widehat{s}(X) \tag{1}$$

which defines a nontrivial two-cocycle on the Lie algebra $\mathcal{G}$:

$$\omega(X,Y) \equiv -\lambda\,\langle \widehat{s}(X) \mid Y\rangle \quad \forall\, X,Y \in \mathcal{G} \tag{2}$$

where λ is a model-dependent normalization constant. The Jacobi identity (1) can be integrated ($Y \to g = \exp Y$) to get a unique nontrivial $\mathcal{G}^*$-valued group one-cocycle $S(g)$ in terms of the Lie-algebra cocycle operator $\widehat{s}$ (provided $H^1(G) = \varnothing$, $\dim H^2(G) = 1$; see [19]):

$$\mathrm{ad}^*(X)\,S(g) = \mathrm{Ad}^*(g)\,\widehat{s}\big(\mathrm{Ad}(g^{-1})\,X\big) - \widehat{s}(X) \quad \forall X \in \mathcal{G} \tag{3}$$

satisfying the relations

$$\widehat{s}(X) = \frac{d}{d\tau} S\big(e^{\tau X}\big)\bigg|_{\tau=0}, \qquad S(g_1 g_2) = S(g_1) + \mathrm{Ad}^*(g_1)\,S(g_2), \tag{4}$$

Now, we can easily generalize the adjoint and coadjoint actions of G and $\mathcal{G}$ to the case with a central extension (acting on elements (X,n), $(Y,m) \in \widetilde{\mathcal{G}}$ and $(U,c) \in \widetilde{\mathcal{G}}^*$; see, e.g., [20]):

$$\widetilde{\mathrm{Ad}}(g)\,(X,n) = \big(\mathrm{Ad}(g)\,X,\ n + \lambda\,\langle S(g^{-1}) \mid X\rangle\big) \ , \tag{5}$$

$$\widetilde{\mathrm{ad}}(X,n)\,(Y,m) \equiv [(X,n),(Y,m)] = \Big([X,Y], -\lambda\,\langle \widehat{s}(X) \mid Y\rangle\Big) \ , \tag{6}$$

$$\begin{aligned} \widetilde{\mathrm{Ad}}^*(g)\,(U,c) &= \big(\mathrm{Ad}^*(g)\,U + c\lambda\,S(g),\ c\big) \ , \\ \widetilde{\mathrm{ad}}^*(X,n)\,(U,c) &= \big(\mathrm{ad}^*(X)\,U + c\lambda\,\widehat{s}(X),\ 0\big) \ . \end{aligned} \tag{7}$$

Also, the bilinear form $\langle\cdot \mid \cdot\rangle$ on $\mathcal{G}^* \otimes \mathcal{G}$ can be extended to a bilinear form on $\widetilde{\mathcal{G}}^* \otimes \widetilde{\mathcal{G}}$ as

$$\big\langle (U,c) \mid (X,n)\big\rangle = \langle U \mid X\rangle + cn \ . \tag{8}$$

The physical interpretation of the $\mathcal{G}$-cocycle $\widehat{s}$ is that of "anomaly" of the Lie algebra [i.e., existence of a c-number term in the commutator (6)], whereas the group cocycle $S(g)$ is the integrated "anomaly," i.e., the "anomaly" for finite group transformations [see eqs. (5) and (4)].

Another basic geometric object is the fundamental $\mathcal{G}$-valued Maurer–Cartan one-form $Y(g)$ on G satisfying $dY(g) = \frac{1}{2}\left[Y(g), Y(g)\right]$. Here and in what follows d denotes the exterior derivative. $Y(g)$ is related to the group one-cocycle $S(g)$ through the equation

$$dS(g) = \mathrm{ad}^*\big(Y(g)\big)\, S(g) + \widehat{s}\big(Y(g)\big) \tag{9}$$

and possesses a group one-cocycle property similar to that of $S(g)$ (4):

$$Y(g_1 g_2) = Y(g_1) + \mathrm{Ad}(g_1)\, Y(g_2) \ . \tag{10}$$

The group- and algebra-cocycles $S(g)$ and $\widehat{s}(X)$ can be generalized to include trivial (co-boundary) parts [(U_0, c) being an arbitrary point in the extended dual space $\widetilde{\mathcal{G}}^*$]:

$$\Sigma(g) \equiv \Sigma\Big(g; (U_0, c)\Big) = c\lambda S(g) + \mathrm{Ad}^*(g)\, U_0 - U_0 \ , \tag{11}$$

$$\widehat{\sigma}(X) \equiv \widehat{\sigma}\big(X; (U_0, c)\big) = \mathrm{ad}^*(X)\, U_0 + c\lambda\, \widehat{s}(X) = \frac{d}{d\tau}\Sigma\big(e^{\tau X}\big)\Big|_{\tau=0} \ . \tag{12}$$

The generalized cocycles (11) and (12) satisfy the same relations as (4), (9), and (3).

2.2. Coadjoint Orbits. The coadjoint orbit of G, passing through the point (U_0, c) of the dual space $\widetilde{\mathcal{G}}^*$, is defined as [cf.(7)]:

$$\mathcal{O}_{(U_0,c)} \equiv \left\{ \Big(U(g), c\Big) \in \widetilde{\mathcal{G}}^*; \quad U(g) = U_0 + \Sigma(g) = \mathrm{Ad}^*(g)\, U_0 + c\lambda S(g) \right\} \ . \tag{13}$$

The orbit (13) is a right coset $\mathcal{O}_{(U_0,c)} \simeq G/G_{\mathrm{stat}}$, where G_{stat} is the stationary subgroup of the point (U_0, c) w.r.t. the coadjoint action (7):

$$G_{\mathrm{stat}} = \left\{ k \in G; \quad \Sigma(k) \equiv c\lambda S(k) + \mathrm{Ad}^*(k)\, U_0 - U_0 = 0 \right\} \ . \tag{14}$$

The Lie algebra corresponding to G_{stat} is

$$\mathcal{G}_{\mathrm{stat}} \equiv \left\{ X_0 \in \mathcal{G}; \quad \widehat{\sigma}(X_0) \equiv \mathrm{ad}^*(X_0)\, U_0 + c\lambda\, \widehat{s}(X_0) = 0 \right\} \ . \tag{15}$$

The physical meaning of $\mathcal{G}_{\mathrm{stat}}$ is that of a maximal "anomaly-free" subalgebra of $\mathcal{G}$, i.e., the maximal subalgebra on which the central extension vanishes [cf.(5)–(7)].

Now, using the basic geometric objects from Section 2.1, we can express the Kirillov–Kostant symplectic form Ω_{KK} [18] on $\mathcal{O}_{(U_0,c)}$ for any infinite-dimensional (centrally extended) group G in a simple compact form [20] . Namely, introducing the centrally extended objects

$$\widetilde{\Sigma}(g) \equiv (\Sigma(g), c) \in \widetilde{\mathcal{G}}^*, \qquad \widetilde{Y}(g) \equiv (Y(g), m_Y(g)) \in \widetilde{\mathcal{G}} \ , \tag{16}$$

$$d\widetilde{\Sigma}(g) = \widetilde{\mathrm{ad}}^*\big(\widetilde{Y}(g)\big)\, \widetilde{\Sigma}(g), \qquad d\widetilde{Y}(g) = \frac{1}{2}\left[\widetilde{Y}(g), \widetilde{Y}(g)\right] \tag{17}$$

we obtain [using (8) and (9)]:

$$\Omega_{KK} = -d\Big(\Big\langle \widetilde{\Sigma}(g) \mid \widetilde{Y}(g) \Big\rangle\Big) = -\frac{1}{2}\Big\langle d\widetilde{\Sigma}(g) \mid \widetilde{Y}(g) \Big\rangle \ . \tag{18}$$

2.3. Geometric Actions and Symmetries. The geometric action on a coadjoint orbit $\mathcal{O}_{(U_0,c)}$ of an arbitrary infinite-dimensional (centrally extended) group G can now be written compactly as [20, 21]

$$\widetilde{W}[g] = \int d^{-1}\Omega_{KK} - \int dt\, H = -\int \left\langle \widetilde{\Sigma}(g) \mid \widetilde{Y}(g) \right\rangle - \int dt\, H\left[\widetilde{\Sigma}(g)\right] \; . \tag{19}$$

In more detail, introducing the explicit expressions (16), (17), (11), and (12), the "kinetic" part of (19) reads

$$\begin{aligned} W[g] = &\int \left\langle U_0 \mid Y(g^{-1}) \right\rangle \\ &- c\lambda \int \left[\left\langle S(g) \mid Y(g) \right\rangle - \frac{1}{2} d^{-1} \left(\left\langle \widehat{s}(Y(g)) \mid Y(g) \right\rangle \right) \right] \; . \end{aligned} \tag{20}$$

The integral in (19), (20) is over the one-dimensional curve on the phase space $\mathcal{O}_{(U_0,c)}$ with a "time-evolution" parameter t . Along the curve the exterior derivative becomes $d = dt\, \partial_t$ and the projection of the one-form $Y(g)$ is $Y(g) = dt\, Y_t(g)$. Note also the presence of the multi-valued (in general) d^{-1} term [22] on the r.h.s. of (20).

The group cocycle properties of $S(g)$ and $Y(g)$ [eqs.(4) and (10)] imply the following fundamental group composition law [21] [with $\Sigma(g)$ as in (11)]:

$$W[g_1 g_2] = W[g_1] + W[g_2] + \int \left\langle \Sigma(g_2) \mid Y(g_1^{-1}) \right\rangle \; . \tag{21}$$

Equation (21) is a generalization of the well-known Polyakov–Wiegmann composition law [23] in WZNW models to geometric actions on coadjoint orbits of arbitrary groups with central extensions.

The whole symmetry structure of the geometric action (20) is encoded in Eq. (21). Indeed, considering first arbitrary right group translations $g \to g\exp(\tau\varepsilon)$, $\varepsilon \in \mathcal{G}$, we get from (21):

$$\begin{aligned} \frac{d}{d\tau} W\left[g\, \exp(\tau\varepsilon)\right]\Big|_{\tau=0} \equiv \delta^R_\varepsilon W[g] &= \int \left\langle \widehat{\sigma}(\varepsilon) \mid Y(g^{-1}) \right\rangle \\ &= -\int \left\langle \widehat{\sigma}\left(Y(g^{-1})\right) \mid \varepsilon \right\rangle \; . \end{aligned} \tag{22}$$

Recalling (15) one finds "gauge" invariance of $W[g]$ under right group translations from the stationary ("anomaly-free") subgroup G_{stat} (14) of the orbit $\mathcal{O}_{(U_0,c)}$ (13): $\delta^R_{\varepsilon_0} W[g] = 0$ for arbitrary time-dependent $\varepsilon_0(t) \in \mathcal{G}_{\mathrm{stat}}$ (15). This fact demonstrates the geometric meaning of "hidden" local symmetries [24] in models with arbitrary infinite-dimensional Noether symmetry groups.

Next, for arbitrary left group translations $g \to \exp(\tau\varepsilon)\, g$ one obtains, using (21), the Noether theorem

$$\frac{d}{d\tau} W\left[\exp(\tau\varepsilon)\, g\right]\Big|_{\tau=0} \equiv \delta^L_\varepsilon W[g] = -\int \left\langle \Sigma(g) \mid d\varepsilon \right\rangle \; , \tag{23}$$

i.e., $\Sigma(g)$ (11) is a Noether conserved current when the Hamiltonian is absent in (19): $\partial_t \Sigma(g) = 0$. More generally, one gets the equations of motion

$$\partial_t \Sigma(g) + \mathrm{ad}^*\left(\frac{\delta H}{\delta \Sigma(g)}\right) \Sigma(g) + \widehat{\sigma}\left(\frac{\delta H}{\delta \Sigma(g)}\right) = 0 \tag{24}$$

using notations (11), (12).

3.1. Geometric Actions and Ward Identities. Let us consider the Legendre transform of the geometric action $W[g]$ (20) on a coadjoint orbit of G:

$$\Gamma[g] \equiv W[g] - \int \left\langle \Sigma(g) \mid \frac{\delta W[g]}{\delta \Sigma(g)} \right\rangle = W[g] + \int \langle \Sigma(g) \mid Y(g) \rangle = -W[g^{-1}] \ . \tag{25}$$

One can easily show that, as a functional of $y \equiv Y_t(g)$, (25) satisfies the functional equation

$$\partial_t \frac{\delta \Gamma}{\delta y} - \mathrm{ad}^*(y) \frac{\delta \Gamma}{\delta y} - \widehat{\sigma}(y) = 0 \ . \tag{26}$$

As shown in [21, 25], Eq. (26) coincides with the renormalized Ward identity for the following functional integral:

$$\exp i\Gamma[y] = \int \mathcal{D}\Phi \exp i\Big\{ W_0[\Phi] + \int \langle J(\Phi) \mid y \rangle \Big\} \ . \tag{27}$$

The notations in (27) are as follows. $W_0[\Phi]$ is a classical action of "matter" fields Φ_a possessing an infinite-dimensional Noether symmetry group G_0 with generators $\{T^I\}$. The index I is a short-hand notation for $I = \big((x_1, \ldots, x_p); A\big)$, including, in general, both continuous parameters $(x_1, \ldots, x_p)$ (e.g., in the case of p-brane models) as well as discrete indices A (as in the case of Kac–Moody groups). The Noether group G_0 is, in general, a contraction ("classical" limit) of G . The corresponding Noether conserved currents $J^I(\Phi) = T_I^* \, J^I(\Phi)$, which belong to the dual space $\mathcal{G}_0^* \simeq \mathcal{G}^*$, span the Poisson bracket algebra of the form

$$\{ J^I(\Phi), J^K(\Phi) \}_{PB} = -\mathring{\omega}^{IK} - \mathring{\omega}^{IK}_L J^L(\Phi) \ , \tag{28}$$

where $\mathring{\omega}^{IK}_L$ and $\mathring{\omega}^{IK}$ denote the structure constants and the (possible) central extension of the "classical" Noether Lie algebra $\mathcal{G}_0$ w.r.t. the basis $\{T^I\}$.

Thus, the Legendre transform of the G co-orbit action (25) is the exact solution for the quantum effective action $\Gamma[y]$ (27) no matter what the specific form of the classical "matter" action $W_0[\Phi]$, provided the Noether symmetries of the latter form a group G_0 coinciding with, or being contraction of, G.

3.2. Examples of Geometric Actions.

3.2.1. *Kac–Moody Groups.* The Kac–Moody group elements $g \simeq g(x)$ are smooth mappings $S^1 \to G_0$, where G_0 is a finite-dimensional Lie group with generators $\{T^A\}$. The explicit form of (5)–(7) reads in this case:

$$\begin{aligned}
\mathrm{Ad}(g)\,X &= g(x)X(x)g^{-1}(x), & \mathrm{ad}(X_1)\,X_2 &= \big[X_1(x), X_2(x)\big] \ , \\
\mathrm{Ad}^*(g)\,U &= g(x)U(x)g^{-1}(x), & \mathrm{ad}^*(X)\,U &= \big[X(x), U(x)\big] \ , \\
X_{1,2}(x) &= X^A_{1,2}(x)T_A, & U(x) &= U_A(x)T^A, \\
\widehat{s}(X) &= \partial_x X(x), & S(g) &= \partial_x g(x)\,g^{-1}(x) \ , \\
Y(g) &= dg(x)\,g^{-1}(x) \ .
\end{aligned} \tag{29}$$

Plugging (29) into (20) one obtains the well-known WZNW action for G_0-valued chiral fields coupled to an external "potential" $U_0(x)$.

3.2.2. *Virasoro Group.* The Virasoro group elements $g \simeq F(x)$ are smooth diffeomorphisms of the circle S^1. Group multiplication is given by composition of diffeomorphisms in inverse order: $g_1 \cdot g_2 =$

$F_2 \circ F_1(x) = F_2(F_1(x))$. Equations (5)–(7) now have the following explicit form:

$$
\begin{aligned}
\mathrm{Ad}(F)\,X &= \left(\partial_x F\right)^{-1} X\left(F(x)\right) \;, \\
\mathrm{Ad}^*(F)\,U &= \left(\partial_x F\right)^{2} U\left(F(x)\right) \;, \\
\mathrm{ad}\,(X)Y \equiv [X,Y] &= X\partial_x Y - (\partial_x X)Y \;, \\
\mathrm{ad}^*(X)\,U &= X\partial_x U + 2(\partial_x X)U \;, \\
S(F) &= \frac{\partial_x^3 F}{\partial_x F} - \frac{3}{2}\left(\frac{\partial_x^2 F}{\partial_x F}\right)^2 \;, \\
Y(F) &= \frac{dF}{\partial_x F}, \qquad \hat{s}(X) = \partial_x^3 X \;.
\end{aligned}
\tag{30}
$$

Here $S(F)$ is the well-known Schwarzian. Plugging (30) into the general expressions (20) and (21) one reproduces the well-known Polyakov $D = 2$ induced gravity action (coupled to an external stress-tensor $U_0(x)$):

$$
W[F] = \int dt\,dx\,\left[-U_0\left(F(t,x)\right)\partial_x F\,\partial_t F + \frac{c}{48\pi}\frac{\partial_t F}{\partial_x F}\left(\frac{\partial_x^3 F}{\partial_x F} - 2\frac{(\partial_x^2 F)^2}{(\partial_x F)^2}\right)\right]
\tag{31}
$$

and its group composition law [24, 26, 27].

3.2.3. $(\mathbf{N},\mathbf{0})$ $\mathbf{D=2}$ *Super-Virasoro Group* ($\mathbf{N}\leq 4$). Here it is appropriate to use the manifestly $(N,0)$ supersymmetric formalism. The points of the $(N,0)$ superspace are labeled as (t,z), $z\equiv(x,\theta^i)$, $i=1,\,\ldots\,,N$. The group elements are given by superconformal diffeomorphisms

$$
z \equiv (x,\theta^j) \rightarrow \widetilde{Z} \equiv \left(F(x,\theta^j),\widetilde{\Theta}^i(x,\theta^j)\right)
\tag{32}
$$

obeying the superconformal constraints:

$$
\begin{aligned}
&D^j F - i\widetilde{\Theta}^k D^j\widetilde{\Theta}_k = 0, \quad D^j\widetilde{\Theta}^l D^k\widetilde{\Theta}_l - \delta^{jk}\left[D\widetilde{\Theta}\right]_N^2 = 0 \;, \\
&\left[D\widetilde{\Theta}\right]_N^2 \equiv \frac{1}{N} D^m\widetilde{\Theta}^n D_m\widetilde{\Theta}_n \;.
\end{aligned}
\tag{33}
$$

Here the following superspace notations are used:

$$
D^i = \frac{\partial}{\partial\theta_i} + i\theta^i\partial_x, \qquad D^N \equiv \frac{1}{N!}\epsilon_{i_1\ldots i_N} D^{i_1}\ldots D^{i_N} \;.
\tag{34}
$$

The $(N,0)$ supersymmetric analogues of (30) read

$$
\begin{aligned}
\mathrm{Ad}(\widetilde{Z})\,X &= \left(\left[D\widetilde{\Theta}\right]_N^2\right)^{-1} X(\widetilde{Z}(z)) \;, \\
\mathrm{Ad}^*(\widetilde{Z})\,U &= \left(\left[D\widetilde{\Theta}\right]_N^2\right)^{2-\frac{N}{2}} U(\widetilde{Z}(z)) \;,
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
[X,Y] &= X\partial_x Y - (\partial_x X)Y - \frac{i}{2}D_k X\,D^k Y \;, \\
\mathrm{ad}^*(X)\,U &= X\partial_x U + (2-\tfrac{N}{2})(\partial_x X)U - \frac{i}{2}D_k X D^k U \;,
\end{aligned}
\tag{36}
$$

$$
\hat{s}_N(X) = i^{N(N-2)} D^N \partial_x^{3-N} X, \qquad Y_N(\widetilde{Z}) = \left(dF + i\widetilde{\Theta}^j d\widetilde{\Theta}_j\right)\left(\left[D\widetilde{\Theta}\right]_N^2\right)^{-1} \;.
\tag{37}
$$

The associated $\mathcal{G}^*$-valued group one-cocycles $S_N(\widetilde{Z})$ coincide with the well-known [28] $(N,0)$ super-Schwarzians. Inserting the latter and (37) into (20), one obtains the WZNW action of induced $(N,0)$ $D=2$ supergravity (i.e., the $(N,0)$ supersymmetric generalization of the Polyakov $D=2$ gravity action (31) for any $N \leq 4$) [20, 29]:

$$W_N[\widetilde{Z}] = \int dt\,(dz) \left[\partial_t \left(\ln \left[D\widetilde{\Theta}\right]^2_N\right) D^N \partial_x^{1-N} \left(\left[D\widetilde{\Theta}\right]^2_N\right) - U_0(\widetilde{Z})\left(\left[D\widetilde{\Theta}\right]^2_N\right)^{1-\frac{N}{2}} \left(\partial_t F + i\widetilde{\Theta}^j \partial_t \widetilde{\Theta}_j\right)\right] . \tag{38}$$

4. APPLICATIONS TO INDUCED $\mathbf{W}_\infty$-GRAVITY AND KP HIERARCHY

4.1. Deformations of Algebras of Area-Preserving Diffeomorphisms. Let us now concentrate on the algebras $\mathbf{w}_\infty$ of area-preserving diffeomorphisms on two-dimensional surfaces. As shown in [30], the family of possible deformations $\mathbf{W}_\infty(q)$ of the initial "classical" $\mathbf{w}_\infty$ depends on a single parameter q and, for each fixed value of q, $\mathbf{W}_\infty(q)$ possesses a one-dimensional cohomology with values in $\mathbb{R}$. In particular, for $q=1$ one finds that $\mathbf{W}_\infty(1) \simeq \widetilde{\mathcal{DOP}(S^1)}$ – the centrally extended algebra of differential operators on the circle, which was recently studied in [7]. The equivalence of $\widetilde{\mathcal{DOP}(S^1)}$ to the original definition of $\mathbf{W}_\infty(1)$ [3, 5] was explicitly demonstrated in [31].

More precisely, let us consider the following class of infinite-dimensional Lie algebras $\mathcal{G} = \mathcal{DOP}(S^1)_{\geq M}$, $M = 0, 1, 2, \ldots$, of symbols of differential operators[1] on the circle S^1 [8]:

$$\mathcal{DOP}(S^1)_{\geq M} = \left\{ X \equiv X(\xi, x) = \sum_{k \geq M} \xi^k X_k(x) \right\} . \tag{39}$$

$\mathcal{DOP}(S^1)_{\geq 1}$ is isomorphic to the "ordinary" $\mathbf{W}_\infty$ algebra, whereas $\mathcal{DOP}(S^1)_{\geq 0}$ is isomorphic to the $\mathbf{W}_{1+\infty}$ algebra.

For any pair $X,\, Y \in \mathcal{G} = \mathcal{DOP}(S^1)_{\geq M}$ the Lie commutator is given in terms of the associative (and *non*commutative) symbol product, denoted henceforth by a circle $\circ$[2]:

$$[X,Y] \equiv X \circ Y - Y \circ X; \qquad X \circ Y \equiv X(\xi,x)\, \exp\left(\overleftarrow{\partial_\xi}\, \overrightarrow{\partial_x}\right)\, Y(\xi,x) . \tag{40}$$

The dual space $\mathcal{G}^* = \mathcal{DOP}^*(S^1)_{\geq M}$ is defined by factoring space $\Psi\mathcal{DO}(S^1) = \left\{ U(\xi,x) = \sum_{k=1}^{\infty} \xi^{-k} U_k(x) \right\}$ of all purely pseudodifferential symbols [32] on S^1 modulo the space of "zero" pseudodifferential symbols $U_{\leq M}(\xi,x)$ (cf. [33])

$$\mathcal{G}^* = \left\{ U_* \,; U_*(\xi,x) = U(\xi,x) - U_{\leq M}(\xi,x) \quad \text{for} \quad \forall U \in \Psi\mathcal{DO} \right\} , \tag{41}$$

$$U_{\leq M}(\xi,x) = e^{\partial_x \partial_\xi} \left(\sum_{k=1}^{M} \xi^{-k} \sum_{l=1}^{k} \binom{k-1}{l-1} \partial_x^{k-l} U_l(x) \right) \tag{42}$$

with the natural bilinear form

$$\langle U \mid X \rangle \equiv \int dx\, \operatorname*{Res}_\xi U \circ X = \int dx\, \operatorname*{Res}_\xi \left(e^{-\partial_x \partial_\xi}\, U(\xi,x) \right) X(\xi,x) . \tag{43}$$

[1] Let us recall [32] the correspondence between (pseudo)differential operators and symbols: $X(\xi,x) = \sum_k \xi^k X_k(x) \longleftrightarrow \widehat{X} = \sum_k X_k(x) \partial_x^k$.

[2] The coefficients in the ξ-expansion of the symbol product $X \circ Y = \sum_{k \geq M} (X \circ Y)_k(x)\, \xi^k$ are given by infinite series. In order to secure convergence, one might consider rescaling of the differential operator symbol ξ by a small parameter h so that (taking, e.g., $M=0$) $(X \circ Y)_k(x) = \sum_0^\infty h^n X_n(x) \left\{ \sum_0^{\min(k,n)} h^{k-m} \binom{n}{m} \partial_x^{n-m} Y_{k-m}(x) \right\}$.

In particular, according to (43) any "zero" pseudodifferential symbol of the form (42) is "orthogonal" to any differential symbol $X \in \mathcal{DOP}(S^1)_{\geq M}$, i.e., $\langle U_{\leq M} \mid X\rangle = 0$. Having the bilinear form (43) one can define the coadjoint action of $\mathcal{DOP}(S^1)_{\geq M}$ on $\mathcal{DOP}^*(S^1)_{\geq M}$ via

$$\left(\mathrm{ad}^*(X)\,U\right)(\xi,x) \equiv [\,X,U\,]_* = [\,X,U\,]_- - [\,X,U\,]_{\leq M} \; . \tag{44}$$

Here and in what follows, the subscript $(-)$ indicates taking that part of the symbol containing all negative powers in the ξ-expansion, whereas the subscript $*$ indicates projecting the symbol on the dual space (41).

The central extension in $\widetilde{\mathcal{G}} \equiv \widetilde{\mathcal{DOP}(S^1)}_{\geq M} = \mathcal{DOP}(S^1)_{\geq M} \oplus \mathbb{R}$ is given by the two-cocycle $\omega(X,Y) = -\frac{1}{4\pi}\langle \widehat{s}(X) \mid Y\rangle$, where the cocycle operator $\widehat{s} : \mathcal{G} \to \mathcal{G}^*$ explicitly reads [7] (cf. also [33]):

$$\widehat{s}(X) = [\,X, \ln\xi\,]_* \; . \tag{45}$$

Let us now consider the Lie group $G = DOP(S^1)_{\geq M}$ defined as exponentiation of the Lie algebra $\mathcal{G} = \mathcal{DOP}(S^1)_{\geq M}$ (39):

$$G = \left\{ g(\xi,x) = \mathrm{Exp}\, X(\xi,x) \equiv \sum_{N=0}^{\infty} \frac{1}{N!} \underbrace{X(\xi,x) \circ \cdots \circ X(\xi,x)}_{\text{N times}} \right\} \; , \tag{46}$$

and the group multiplication is just the symbol product $g \circ h$. The adjoint and coadjoint action of $G = DOP(S^1)_{\geq M}$ on the Lie algebra $\mathcal{DOP}(S^1)_{\geq M}$ and its dual space $\mathcal{DOP}^*(S^1)_{\geq M}$, respectively, is given as:

$$\mathrm{Ad}(g)\,X = g \circ X \circ g^{-1}; \qquad \mathrm{Ad}^*(g)\,U = (g \circ X \circ g^{-1})_* \; . \tag{47}$$

4.2. WZNW Action for Induced $\mathbf{W}_\infty$ Gravity. After these preliminaries it is easy to write the explicit expressions of the two fundamental objects $S(g)$ (4) (the nontrivial $\mathcal{G}^*$-valued one-cocycle on the group G, or the "anomaly" for finite group transformations) and the Maurer–Cartan form $Y(g)$ interrelated through Eq. (9), which enter the construction of the geometric action on a coadjoint orbit of $G = DOP(S^1)_{\geq M}$:

$$S(g) = -\Big(\,[\,\ln\xi, g(\xi,x)\,] \circ g^{-1}(\xi,x)\Big)_*; \qquad Y(g) = dg(\xi,x) \circ g^{-1}(\xi,x) \; . \tag{48}$$

Plugging (48) and (45) in the general formula (20), one obtains the co-orbit geometric action (the explicit dependence of symbols on $(\xi,x;t)$ will, in general, be suppressed below):

$$\begin{aligned} W_{DOP(S^1)_{\geq M}}[g] = &-\int dt\,dx\, \operatorname*{Res}_{\xi} U_0 \circ g^{-1} \circ \partial_t g \\ &+ \frac{c}{4\pi}\int\!\!\int dx\, \operatorname*{Res}_{\xi}\Big([\,\ln\xi, g\,] \circ g^{-1} \circ \partial_t g \circ g^{-1} \\ &- \frac{1}{2} d^{-1}\{ [\,\ln\xi, dg \circ g^{-1}\,] \wedge (dg \circ g^{-1})\}\Big) \; . \end{aligned} \tag{49}$$

According to the discussion in Section 3, the Legendre transform $\Gamma[g] = -W[g^{-1}]$ of (49) is precisely the WZNW anomalous effective action of induced $\mathbf{W}_\infty$ or $\mathbf{W}_{1+\infty}$ gravity for $M = 1$, 0, respectively. The physical meaning of the first term on the r.h.s. of (49) is that of coupling of the chiral $\mathbf{W}_{(1)+\infty}$ Wess–Zumino field $g = g(\xi,x;t)$ to a chiral $\mathbf{W}_{(1)+\infty}$ gravity "background." From the general formula (21) we get the following fundamental group composition law for $\mathbf{W}_{(1)+\infty}$ gravity action:

$$\begin{aligned} W[g \circ h] = &\; W[g] + W[h] \\ &+ \int dt\,dx\, \operatorname*{Res}_{\xi}\Big\{ \Big(h \circ U_0 \circ h^{-1} \\ &- \frac{c}{4\pi}[\,\ln\xi, h\,] \circ h^{-1}\Big) \circ g^{-1} \circ \partial_t g \Big\} \; . \end{aligned} \tag{50}$$

The action (49) implies the Poisson brackets

$$\left\{ S[g](\xi, x),\, S[g](\eta, y) \right\}_{PB} = -\frac{4\pi}{c} \left[S[g](\xi, x) + \ln \xi,\ \delta_{DOP}(y, \eta;\ x, \xi) \right]_* \ , \tag{51}$$

where the symbol commutator on the r.h.s. is the projected one as in (44), and $\delta_{DOP}(\cdot\,;\,\cdot) \in \mathcal{DOP}^*(S^1)_{\geq M} \otimes \mathcal{DOP}(S^1)_{\geq M}$ denotes the kernel of the δ-function on the space of differential operator symbols:

$$\delta_{DOP}(x, \xi;\ y, \eta) = e^{\partial_x \partial_\xi} \left(\sum_{k=M}^{\infty} \xi^{-(k+1)}\, \eta^k\, \delta(x-y) \right) \ . \tag{52}$$

Equation (51) is a compact expression of the Poisson-bracket realization of $\mathcal{DOP}(S^1)_{\geq M}$, in particular, for $\mathbf{W}_{(1)+\infty} \simeq \mathcal{DOP}(S^1)_{\geq 0,1}$. The component fields $S_r(x)$ in the ξ-expansion of the pseudodifferential symbol $S[g](\xi, x) = \sum_{r \geq M+1} S_r(x)\, \xi^{-r}$ turn out to be quasi-primary conformal fields of spin $r \geq M$. The genuine primary fields $\mathcal{W}_r(x)$ are obtained from $S_r(x)$ by adding derivatives of the lower spin fields $S_q(x)$, $q \leq r-1$.

At this point it would be instructive to explicate formulas (45) and (48) when the elements of $G = \mathcal{DOP}(S^1)_{\geq 1}$ and $\mathcal{G} = \mathcal{DOP}(S^1)_{\geq 1} \simeq \mathbf{W}_\infty$ are restricted to the Virasoro subgroup (subalgebra, respectively):

$$\begin{aligned} X(\xi, x) = \xi\, \omega(x) &\longleftrightarrow \omega(x)\partial_x \in \mathcal{V}ir \\ g(\xi, x) = \mathrm{Exp}\big(\xi\, \omega(x)\big) &\longleftrightarrow F(x) \equiv \exp\big(\omega(x)\partial_x\big)\, x \in \mathrm{Diff}(S^1) \ . \end{aligned} \tag{53}$$

Substituting (53) into (45) and (48), one obtains [cf. eq.(30)]:

$$\begin{aligned} Y(g)\Big|_{g(\xi,x)=\mathrm{Exp}\left(\xi\,\omega(x)\right)} &= \xi \frac{dF(x)}{\partial_x F(x)} \ ; \\ \widehat{s}(X) = \left[\xi \omega(x), \ln \xi \right]_* &= -\frac{1}{6} \xi^{-2} \partial_x^3 \omega(x) + \cdots \ , \\ S(g)\Big|_{g(\xi,x)=\mathrm{Exp}\left(\xi\,\omega(x)\right)} &= -\frac{1}{6}\, \xi^{-2} \left(\frac{\partial_x^3 F}{\partial_x F} - \frac{3}{2} \left(\frac{\partial_x^2 F}{\partial_x F} \right)^2 \right) + \cdots \ . \end{aligned} \tag{54}$$

The dots in (54) indicate higher order terms $O(\xi^k)$, $k \geq 3$, which do not contribute in bilinear forms with elements of $\mathcal{V}ir$ (53). Similar formulas (replacing the factors 1/6 in (54) by 1/3) are obtained for the embedding of the Virasoro algebra in $\mathcal{DOP}(S^1)_{\geq 0} \simeq \mathbf{W}_{1+\infty}$. Now, inserting (54) into Eq. (49) (for $M = 0, 1$) one reproduces (31), i.e., Polyakov's action of induced $D = 2$ gravity [24, 26]. In particular, one finds that the $\mathcal{DOP}(S^1)_{\geq M}$ Maurer–Cartan gauge field $Y_t(g^{-1}) = -g^{-1} \circ \partial_t g$ is a generalization of Polyakov's $D = 2$ gravity gauge field $h_{++} = \partial_t F^{-1} / \partial_x F^{-1}$, where F^{-1} denotes the inverse Virasoro group element.

We refer to [33] for a further analysis of the symmetry properties of the $\mathbf{W}_\infty$ gravity action (49), namely, the "hidden" Kac–Moody symmetry, which turns out to be a specific differential operator symbol realization of $SL(\infty; \mathbb{R})$, and the associated classical Sugawara construction.

4.3. Action for KP Hierarchy. Let us now consider the geometric action of the *centerless* $\mathcal{DOP}(S^1)_{\geq 0} \simeq \mathbf{W}_{1+\infty}$ [i.e., put $c = 0$ in Eq. (49)] with a nontrivial Hamiltonian added [cf. (19)]:

$$\begin{aligned} \widetilde{W}[g] = &- \int dt\, dx\, \mathrm{Res}_\xi \big(U^{(0)} \circ g^{-1} \circ \partial_t g \big) \\ &- \frac{1}{N+1} \int dt\, dx\, \mathrm{Res}_\xi \Big(\xi + \big(g \circ U^{(0)} \circ g^{-1} \big)_- \Big)^{N+1} \ , \end{aligned} \tag{55}$$

where $U^{(0)} \in \Psi\mathcal{DO}(S^1)$ is a fixed pseudodifferential symbol, and the exponent in the last term indicates $(N+1)$-th power w.r.t. symbol product. One can easily show that (55) yields the Hamiltonian equation of motion precisely coinciding with the N-th equation of the KP hierarchy [34] of integrable evolution equations.

Indeed, introducing the notation $U(g) \equiv \left(g \circ U^{(0)} \circ g^{-1}\right)_{-}$, the equation of motion associated with (55) reads according to (24)

$$\partial_t U(g) + \left[\left\{ \left(\xi + U(g)\right)^N \right\}_+, U(g) \right]_- = 0 \ , \tag{56}$$

where the subscript $(+)$ denotes taking the purely differential part (the non-negative ξ-power expansion). Using the simple property $\left[\left\{ \mathcal{F}(\xi + U) \right\}_+, \xi + U \right]_+ = 0$, where U is an arbitrary purely pseudodifferential symbol and $\mathcal{F}$ is an arbitrary (analytic) function, one can immediately rewrite Eq. (56) in the form

$$\partial_t(\xi + U(g)) + \left[\left\{ \left(\xi + U(g)\right)^N \right\}_+, \xi + U(g) \right] = 0 \ . \tag{57}$$

Of course, Eq. (57) is nothing but the N-th equation of the KP hierarchy in the standard Lax form [34], where $L \equiv \xi + U(g) = \xi + \sum_{k\geq 1} U_k[g](x)\, \xi^{-k}$ is the symbol of the corresponding pseudodifferential Lax operator. The identification of (57) as a Hamiltonian equation on a coadjoint orbit of the centerless $\mathbf{W}_{1+\infty}$ was first pointed out in [10].

Having an action (55) might help quantization of the KP model.

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